p-adic Hubbard Trees

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Abstract

In complex dynamics, Hubbard trees offer a combinatorial description of the dynamics of post-critically finite (PCF) polynomials. What are the analogous objects in a non-Archimedean setting: what is a p-adic Hubbard tree? We explore this question by studying the critical orbit trees associated to quadratic maps $f_c(z) = z^2 + c$, with $c \in \mathbb{Z}_p$ (for $p > 2$).

Complex Setting

The Hubbard tree H_f of a PCF polynomial f is a finite tree in the Julia set, connecting all points contained in the critical orbits. The action of f on H_f and the embedding of H_f in the complex plane (up to isotopy class) captures all of the important information of the global dynamical system $f : \mathbb{C} \to \mathbb{C}$ (see [2], [8]).

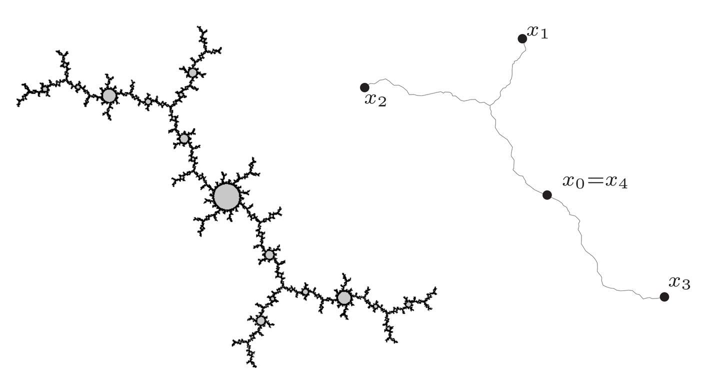

Figure 1: An example of a Hubbard tree from [2], an invariant subset of the Julia set.

Non-Archimedean Setting

We view $\mathbb{Z}_p$ as a subtree of the Berkovich project line over $\mathbb{C}_p$, endowed with the p-adic metric. Each vertex of the tree is a disk with rational radius of the form p^{-n}, denoted $D(a, r)$. Two vertices are connected by an edge (branch) if one disk is contained in the other, so each point has p edges branching off of it. $\mathbb{Z}_p$ is exactly the set of ends of the branches, and the top of the tree is the Gauss point $D(0, 1)$.

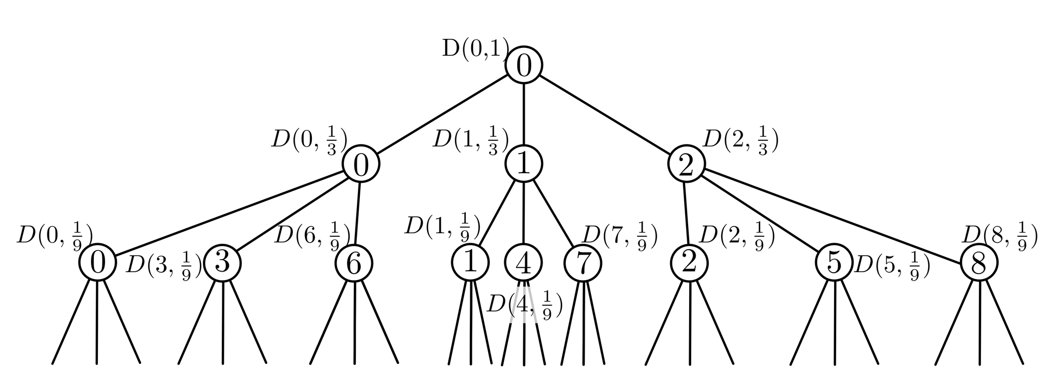

Figure 2: The top of the $\mathbb{Z}_3$ tree.

For $f_c(z) = z^2 + c$, $c \in \mathbb{Z}_p$, we suggest the following Hubbard tree analogy:

> Definition: The **critical orbit tree** for f_c is the convex hull of the critical orbit in the $\mathbb{Z}_p$ tree, together with the induced action of f_c. The **critical orbit tree (mod p)** is the subtree consisting of residue classes (mod p).

We say that a point α has **orbit type $(\mathbf{m}, \mathbf{n})$** if m and n are the least integers such that $f^{m+n}(\alpha) = f^m(\alpha)$. Then m is the tail length and n is the cycle length of the orbit of α.

Main Results

To understand the critical orbit tree structures for f_c, it is necessary to study the behavior of the critical orbit (mod p). The periodic case may be deduced from ([1], [4], [6] and [9]), based on the existence of an attracting n-cycle:

Theorem 1: Periodic (mod p)

Let $p \geq 3$ and consider the critical orbit for $f_c(z) = z^2 + c$, $c \in \mathbb{Z}_p$. If 0 has orbit type $(0, n)$ (mod p), then either 0 is periodic of exact period n or 0 has infinite orbit, with orbit type (m_i, n) (mod p^i) for all $i \geq 1$.

The pre-periodic case builds on the work from [6] and [7] on the length of periodic cycles in $\mathbb{Z}_p$.

Theorem 2: Pre-periodic (mod p)

Let $p \geq 3$ and consider the critical orbit for $f_c(z) = z^2 + c$, $c \in \mathbb{Z}_p$. If 0 is strictly pre-periodic with orbit type (m, n) (mod p), then either 0 has orbit type (m, n) over $\mathbb{Z}_p$ or there exists some $k \geq 1$ in $\mathbb{Z}$ such that

1. 0 has orbit type (m, n) (mod p^i) for all $i \leq k$, and
2. 0 has orbit type $(m, r \cdot n)$ (mod p^j) for all $j > k$, with $r | (p - 1)$.

Otherwise, 0 has infinite orbit, with orbit type (m, n_i) (mod p^i) for all $i \geq 1$.

A key piece of the proof is the fact that if 0 has tail length m (mod p), then the tail length is fixed at m when the critical orbit is calculated modulo higher powers of p, even if the orbit of 0 is infinite over $\mathbb{Z}_p$.

Remark: There is a finite number of PCF parameters in $\mathbb{Z}_p$. The work of [7] and [10] gives a uniform bound on the total number for given prime p.

Theorems 1 and 2 also have the following implications on the tree structures of finite critical orbits in $\mathbb{Z}_p$:

Theorem 3: Critical Orbit Trees in $\mathbb{Z}_p$

Again let $p \geq 3$ and suppose 0 has finite orbit for $f_c(z) = z^2 + c$, $c \in \mathbb{Z}_p$.

1. If 0 is periodic of exact period n, the critical orbit tree coincides with the critical orbit tree (mod p). It is a finite tree with a single vertex of degree n, and $f_c(z)$ acts on the n end points by a cyclic permutation.
2. If 0 is strictly pre-periodic with orbit type (m, n), the critical orbit tree either coincides exactly with the critical orbit tree (mod p) or it differs by one instance of branching.

Remark: Linearization gives a bound on how far into the $\mathbb{Z}_p$ tree the branching can occur, for given prime p, and consequently there is a finite number of possible critical orbit trees for PCF parameters in $\mathbb{Z}_p$.

Examples

In light of Theorem 3, it is straightforward to calculate all possible finite critical orbit trees for a given p. We give the complete list for $p = 3$ and $p = 5$.

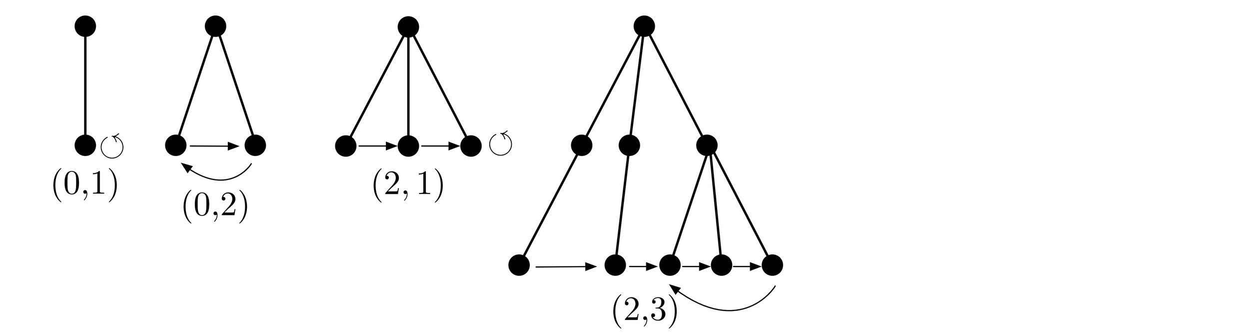

Figure 3: The 4 distinct finite critical orbit trees in $\mathbb{Z}_3$.

Note that the (2,3) tree matches the (2,1) tree (mod 3) and then branches once below (mod 3^2). For $p = 5$ we have an example of 2 distinct critical orbit trees that correspond to orbit type (2,2).

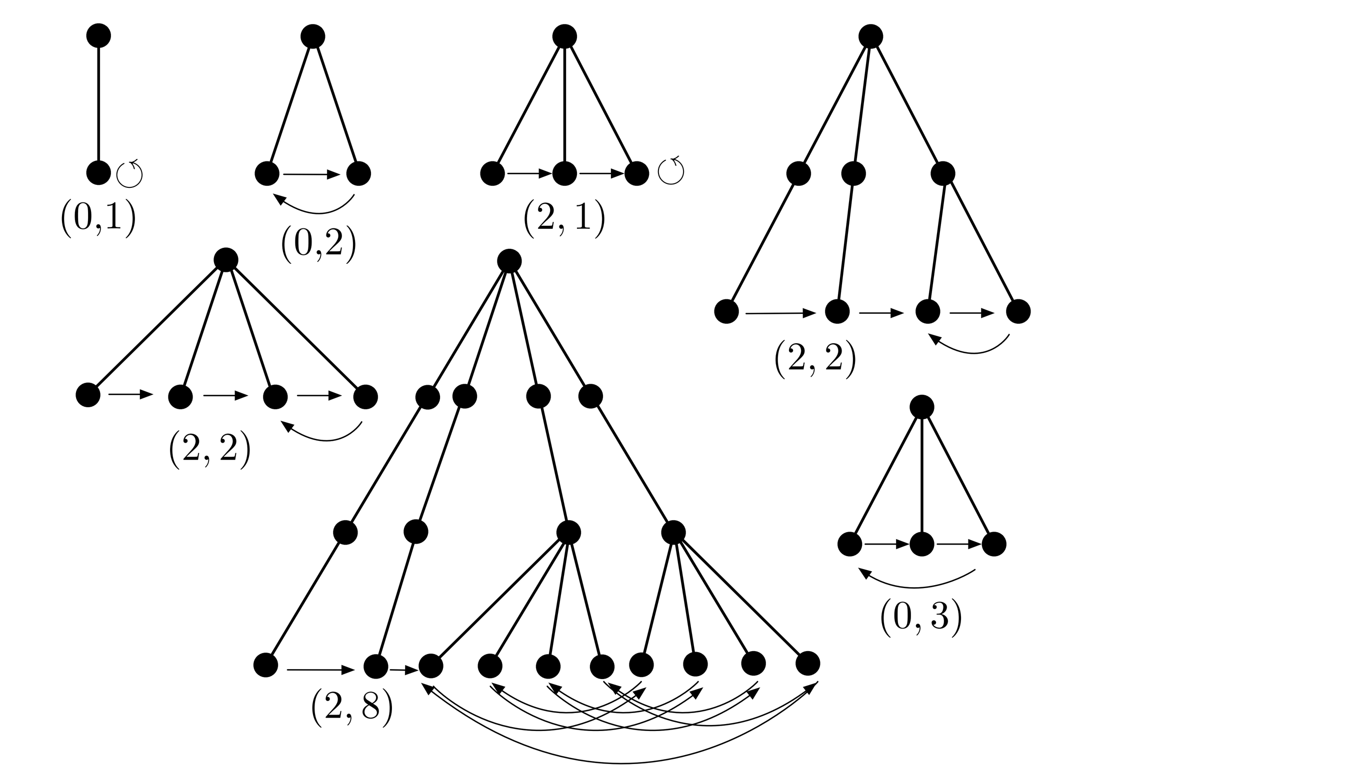

Figure 4: The 7 distinct finite critical orbit trees in $\mathbb{Z}_5$.

Remark: Going beyond the finite orbit trees for $p = 3$, we can give a complete description of the orbit trees for all $c \in \mathbb{Z}_3$ by exploiting the existence of linearization disks near periodic cycles, as detailed in [3] and [5].

References

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